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Probleme de matematica pentru liceu

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$$1+z_k^2=1+\cos^2\frac{(2k+1)\pi}{n}+i\sin\frac{(2k+1)\pi}{n}=2\cos^2\frac{(2k+1)\pi}{n}+$$

$$+2i\sin\frac{(2k+1)\pi}{n}\cos\frac{(2k+1)\pi}{n}=2\cos\frac{(2k+1)\pi}{n}\left[\cos\frac{(2k+1)\pi}{n}+\right.$$

$\left.+i\sin\frac{(2k+1)\pi}{n}\right]$ şi atunci: produsul din stânga, pentru (1), devine:

$$\prod_{k=0}^{n-1}\left(1+z_k^2\right)=2^n\cos\frac{\pi}{n}\cos\frac{3\pi}{n}\cos\frac{5\pi}{n}\cdot\ldots\cdot\cos\frac{(2n-1)\pi}{n}\cdot$$

$$\cdot\left[\cos\frac{1+3+5+\ldots+(2n-1)\pi}{n}+i\sin\frac{1+3+5+\ldots+(2n-1)\pi}{n}\right]=$$

$=2^n\cos\frac{\pi}{n}\cos\frac{3\pi}{n}\cdot\ldots\cdot\cos\frac{(2n-1)\pi}{n}\left[\cos n\pi+i\sin n\pi\right]$, adică am găsit:

$$\prod_{k=0}^{n-1}\left(1+z_k^2\right)=2^n\cos\frac{\pi}{n}\cos\frac{3\pi}{n}\cdot\ldots\cdot\cos\frac{(2n-1)\pi}{n}\cdot\cos n\pi \quad (2).$$

Acum:

$$\prod_{h=1}^{2}\left((x_h)^n+1\right)=\left[(i)^n+1\right]\left[(-i)^n+1\right]=1^n+i^n+(-i)^n+1=\begin{cases}4, & n=4\ell\\ 2, & n=4\ell\pm1\\ 0, & n=4\ell=2\end{cases},$$

$\ell\in\mathbb{N}^*$. Venind cu ultimul rezultat şi cu (2) în (1), obţinem identitatea dorită.

P.303. Polinomul $f=X^2+X+1$ are rădăcinile $x_1=\varepsilon$ şi $x_2=\overline{\varepsilon}$ cu $x_1+x_2=-1$ şi $x_1x_2=1$ şi $x_1^3=x_2^3=1$, iar polinomul $g=Z^n-1$ are $z_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, $k=\overline{0,n-1}$, atunci din lemă:

$$\prod_{k=0}^{n-1}f(z_k)=(-1)^{2n}\prod_{h=1}^{2}g(x_h)\text{, adică }\prod_{k=0}^{n-1}\left(z_k^2+z_k+1\right)=\prod_{h=1}^{2}\left[(x_h)^n-1\right]\ (1).$$

Pentru (1) calculăm, pe rând, cele două produse:

$$1+z_k+z_k^2=1+\cos\frac{4k\pi}{n}+i\sin\frac{4k\pi}{n}+\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}=2\cos^2\frac{2k\pi}{n}+$$

$$+\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}=\cos\frac{2k\pi}{n}\left(2\cos\frac{2k\pi}{n}+1\right)+$$

$+i\sin\frac{2k\pi}{n}\left(2\cos\frac{2k\pi}{n}+1\right)=\left(2\cos\frac{2k\pi}{n}+1\right)\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)=$

$=\left(1+2\cos\frac{2k\pi}{n}\right)z_k$ şi atunci produsul din stânga, pentru (1), devine:

$\prod_{k=0}^{n-1}\left(z_k^2+z_k+1\right)=\prod_{k=0}^{n-1}\left(1+2\cos\frac{2k\pi}{n}\right)\cdot z_k=(-1)^{n+1}\prod_{k=0}^{n-1}\left(1+2\cos\frac{2k\pi}{n}\right)$, adică

am găsit: $(-1)^{n+1}\left(1+2\cos\frac{2\pi}{n}\right)\left(1+2\cos\frac{4\pi}{n}\right)\cdot\ldots\cdot\left(1+2\cos\frac{2(n-1)\pi}{n}\right)=$

$=\prod_{k=0}^{n-1}\left(z_k^2+z_k+1\right)$ (2).

Acum: $\prod_{h=1}^{2}\left(\left(x_h\right)^n-1\right)=\left(\varepsilon^n-1\right)\left(\left(\overline{\varepsilon}\right)^n-1\right)=\left(\varepsilon\overline{\varepsilon}\right)^n-\left(\varepsilon\right)^n-\left(\overline{\varepsilon}\right)^n+1=$

$=\begin{cases}0, & n=3\ell\\ 3, & n=3\ell\pm1\end{cases}$, $\ell\in\mathbb{N}^*$ şi, în final, cu (2) şi (3) în (1), obţinem identitatea dorită.

Cuprins